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Stresses in Anisotropic Nonhomogeneous Cylinders

MANORANJAN MAITI*
Vikram Sarabai Space Center, Trivandrum, India

1. Introduction

THE analysis of stresses in anisotropic nonhomogeneous hollow cylinders is, nowadays, a subject of increasing concern for the design and analysis of solid propellant motors with arbitrary circumferential reinforcement and distribution along the radius. Bieniek et al.[1] evaluated the stresses and deformations in a thick-walled orthotropic cylinder with all material constants proportional to r^m only. In this Note, a general method is developed to find out the stresses and displacement components in an orthotropic nonhomogeneous elastic cylinder with different types of nonhomogeneities introduced in the material constants following the analysis of Wescott.[2]

2. Formulation of the Problem

The equations of equilibrium and compatibility for a circular cylinder in two-dimensional problems are, respectively,

$$d\sigma_r/dr + (\sigma_r - \sigma_\theta)/r = 0 \tag{1}$$

and

$$\varepsilon_r - (r\, d\varepsilon_\theta/dr) - \varepsilon_\theta = 0, \quad \varepsilon_r = du/dr, \quad \varepsilon_\theta = u/r \tag{2}$$

where σ_r, σ_θ are the radial and circumferential stresses, respectively, and u is the only nonvanishing radial displacement component. In the case of plain strain, the preceding equations for the orthotropic nonhomogeneous cylinder with material constants as functions of r only reduce to the following single equation [cf. Bieniek et al.[1]]:

$$\frac{d^3f}{dr^3} + \left(\frac{1}{\alpha_{22}}\frac{d\alpha_{22}}{dr} + \frac{1}{r}\right)\frac{d^2f}{dr^2} + \left(-\frac{\alpha_{11}}{r^2\alpha_{22}} + \frac{1}{r\alpha_{22}}\frac{d\alpha_{12}}{dr}\right)\frac{df}{dr} = 0 \tag{3}$$

where

$$\left.\begin{aligned} \sigma_r &= (1/r)[df(r)/dr], \quad \sigma_\theta = d^2f(r)/dr^2 \\ du/dr &= \alpha_{11}\sigma_r + \alpha_{12}\sigma_\theta, \quad u/r = \alpha_{12}\sigma_r + \alpha_{22}\sigma_\theta \end{aligned}\right\} \tag{4}$$

and α_{11}, α_{12}, α_{22} being the material constants. Hence, the central problem now is to find the stress function $f(r)$ satisfying Eq. (3).

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* Numerical Analyst, Structural Engineering Division.

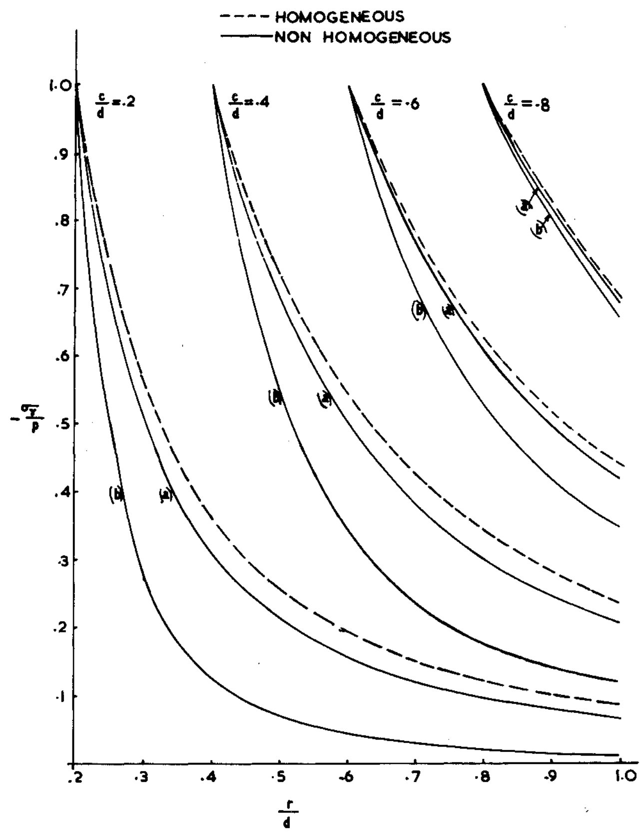

Fig. 1 Radial stresses for cases a and b.

3. Solution

Let

$$\alpha_{11} = \bar{\alpha}_{11}l(r), \quad \alpha_{22} = \bar{\alpha}_{22}m(r) \quad \text{and} \quad \alpha_{12} = \bar{\alpha}_{12}n(r) \tag{5}$$

where $\bar{\alpha}_{11}$, $\bar{\alpha}_{12}$, $\bar{\alpha}_{22}$ are constants. Putting these expressions for α_{11}, α_{12}, α_{22} and changing the variables as

$$\{rm(r)\}^{1/2}\, df/dr = y(r) \tag{6}$$

in Eq. (3), we have

$$d^2y/dr^2 + I(r) = 0 \tag{7}$$

where

$$I(r) = -\frac{\beta}{r^2}\frac{l(r)}{m(r)} + \frac{\alpha}{rm(r)}\frac{dn(r)}{dr} - \frac{1}{2}\frac{dM(r)}{dr} - \frac{1}{4}M^2(r) \tag{8}$$

with

$$\beta = \bar{\alpha}_{11}/\bar{\alpha}_{22}, \quad \alpha = \bar{\alpha}_{12}/\bar{\alpha}_{22} \quad \text{and} \quad M(r) = \frac{1}{m(r)}\frac{dm(r)}{dr} + \frac{1}{r}$$

Now, let us consider the standard transcendental differential equations of the type

$$\frac{d^2w}{dz^2} + p(z)\frac{dw}{dz} + q(z)w = 0 \tag{9}$$

This becomes a) hypergeometric equation for $p = z^{-1}c - A(1-z)^{-1}$, $q = -abz^{-1}(1-z)^{-1}$ with $A = a+b-c+1$ and $w = {}_2F_1(a,b,c;z)$ or $z^{1-c}{}_2F_1(1+a-c, 1+b-c, 2-c; z)$, b) Whittaker's equation for $p = 0$, $q = -\gamma^2/4 + s\gamma z^{-1} + (\frac{1}{4}-t^2)z^{-2}$ with $w = W_{\pm s,t}(\pm\gamma z)$, c) Bessel's equation for $p = z^{-1}$, $q = \pm\gamma^2 - v^2z^{-2}$ with $w = C_v(\gamma z)$, d) Helmholt's equation for $p = 0$, $q = -\gamma^2$ with $w = \exp(\pm\gamma z)$ where ${}_2F_1$, $W_{\pm s,t}$ and C_v represent the transcendental hypergeometric, Whittaker and Bessel functions, respectively. Applying the transformation

$$y = w(z)(dz/dr)^{-1/2}\exp\{\tfrac{1}{2}p(z)(dz/dr)\,dr\} \tag{10}$$

to Eq. (9), we get

$$\frac{d^2y}{dz^2}+\left[S\left(\frac{dz}{dr}\right)+\left(\frac{dz}{dr}\right)^2\left\{-\frac{1}{4}p^2(z)-\frac{1}{2}\frac{dp(z)}{dz}+q(z)\right\}\right]y=0 \quad (11)$$

where

$$S(dz/dr)=\tfrac{1}{2}(d^3z/dr^3)\,dr/dz-\tfrac{3}{4}(d^2z/dr^2)^2(dr/dz)^2 \quad (12)$$

Now, by the replacement of

$$-\tfrac{1}{4}p^2(z)-\tfrac{1}{2}\,dp(z)/dz+q(z) \quad \text{by} \quad L_1f(z)+L_2g(z)+L_3h(z)$$

in Eq. (11), it becomes

$$\frac{d^2y}{dz^2}+\left[S\left(\frac{dz}{dr}\right)+\left(\frac{dz}{dr}\right)^2\{L_1f(z)+L_2g(z)+L_3h(z)\}\right]y=0 \quad (13)$$

The values of L_1, L_2, L_3, $f(z)$, $g(z)$, and $h(z)$ are for

a) hypergeometric equation

$$f(z)=z^{-2},\quad g(z)=(1-z)^{-2},\quad h(z)=z^{-1}(1-z)^{-1},$$
$$L_1=(c/2)-(c^2/4),\quad L_2=(A/2)-(A^2/4),\quad L_3=\tfrac{1}{2}cA-ab$$

b) Whittaker's equation

$$f(z)=1,\quad g(z)=z^{-2},\quad h(z)=z^{-2},\quad L_1=-\gamma^2/4,$$
$$L_2=s\gamma,\quad L_3=\tfrac{1}{4}-t^2$$

c) Bessel's equation

$$f(z)=1,\quad g(z)=0,\quad h(z)=z^{-2},\quad L_1=\pm\gamma^2,$$
$$L_2=0,\quad L_3=\tfrac{1}{4}-v^2$$

d) Helmholtz's equation

$$f(z)=1,\quad g(z)=h(z)=0,\quad L_1=-\gamma^2,\quad L_2=L_3=0$$

Following Eq. (12), we write Eq. (8) as

$$I(r)=-\frac{\beta}{r^2}\frac{l(r)}{m(r)}+\frac{\alpha}{rm(r)}\frac{dn(r)}{dr}-\frac{1}{2}\frac{dM(r)}{dr}-\frac{1}{4}M^2(r)$$
$$=\frac{\beta}{r^2}-\left\{1+\frac{l(r)}{m(r)}\right\}\frac{\beta}{r^2}+\frac{\alpha}{rm(r)}\frac{dn(r)}{dr}+S\left(\frac{dz}{dr}\right)+\left(\frac{dz}{dr}\right)^2S\left(\frac{1}{rm(r)}\frac{dr}{dz}\right)$$

$$\left[\text{by chain rule, } S\left\{\frac{dW(z(r))}{dr}\right\}=S\left(\frac{dz}{dr}\right)+\left(\frac{dz}{dr}\right)^2S\left(\frac{dW}{dz}\right)\right]$$

Putting this value of $I(r)$ in Eq. (7) and then comparing Eqs. (7) and (13), we get

$$\left[\frac{\beta}{r^2}-\left\{1+\frac{l(r)}{m(r)}\right\}\frac{\beta}{r^2}+\frac{\alpha}{rm(r)}\frac{dn(r)}{dr}\right]\left(\frac{dr}{dz}\right)^2+S\left(\frac{1}{rm(r)}\frac{dr}{dz}\right)=L_1f(z)+L_2g(z)+L_3h(z)$$

In order to have the undetermined function $f(r)$ and the expressions for $l(r)$, $m(r)$ and $n(r)$, let us assume

$$(1/r^2)(dr/dz)^2=P_1f(z)+P_2g(z)+P_3h(z) \quad (14)$$

$$S\left(\frac{1}{rm(r)}\frac{dr}{dz}\right)=Q_1f(z)+Q_2g(z)+Q_3h(z) \quad (15)$$

$$\left(\frac{dr}{dz}\right)^2\frac{1}{rm(r)}\frac{dn(r)}{dr}=R_1f(z)+R_2g(z)+R_3h(z) \quad (16)$$

Then, substituting Eqs. (14–16) in Eq. (13) we have

$$\left(\frac{dr}{dz}\right)^2\frac{\beta}{r^2}\left\{1+\frac{l(r)}{m(r)}\right\}=K_1\beta f(z)+K_2\beta g(z)+K_3\beta h(z) \quad (17)$$

with

$$\beta K_i=\beta P_i+Q_i+\alpha R_i-L_i,\quad i=1,2,3 \quad (18)$$

where P_i, Q_i, R_i, K_i, $i=1,2,3$ are all arbitrary multipliers independent of z so that at least one of P_1, P_2, P_3 is nonzero. Giving the different arbitrary values to these multipliers, we get the expressions for $z(r)$ from Eq. (14), $m(r)$ from Eq. (15), $n(r)$ from Eq. (16) and $l(r)$ from Eq. (17). The undetermined arguments and orders of corresponding transcendental function are obtained from Eq. (18). Although the second-order differential equation (15) gives $m(r)$ in terms of two independent functions, we retain only one of them for simplicity.

Hence, knowing the values of all transcendental parameters in terms of the given values P_i, Q_i, R_i, K_i, $i=1,2,3$ we derive the expressions for the material constants α_{11}, α_{12}, α_{22} and the corresponding solutions of Eq. (3) in terms of hypergeometric, Whittaker, Bessel or exponential functions by the help of Eqs. (6) and (10) as

$$\frac{df(r)}{dr}\propto\{rm(r)\}^{-1/2}\left(\frac{dz}{dr}\right)^{-1/2}\exp\left\{\tfrac{1}{2}\int p(z)\frac{dz}{dr}dr\right\}w(z) \quad (19)$$

Having $df(r)/dr$ from Eq. (19), the expressions for the stresses and deformations are obtained by the expressions of Eq. (4). As Eq. (3) is a second-order differential equation in $df(r)/dr$, its solution will have two unknown constants which are evaluated by the boundary conditions of the problem.

4. Evaluation of Stresses

As an example, we consider an anisotropic thick cylinder with some particular types of nonhomogeneities. Let the nonhomogeneous cylinder with outer surface reinforced by a thin elastic casing with inner and outer radii as c and d, respectively, be subjected to internal pressure, P. Then, we get

$$\sigma_r=-P \quad \text{on} \quad r=c \quad (20)$$

and

$$\sigma_r=-[Et/(1-v^2)d]\varepsilon_\theta \quad \text{on} \quad r=d \quad (21)$$

E, v, t are, respectively, elastic modulus, Poisson's ratio and thickness of the elastic casing.

a) Nonhomogeneities based on Bessel's equation: Let

$$P_1=P_2=0,\ P_3=a^{-2};\quad Q_1=Q_2=0,\ Q_3=A(1-A);$$
$$R_1=b^2,\ R_2=R_3=0;\quad K_1=n^2,\ K_2=K_3=0$$

So, we have from Eq. (14)

$$(1/r^2)(dr/dz)^2=(1/a^2)z^{-2} \quad \text{i.e.,} \quad z=A_1r^a$$

from Eq. (15)

$$S\left(\frac{1}{rm(r)}\frac{dr}{dz}\right)=\frac{A(1-A)}{z^2} \quad \text{i.e.,} \quad m(r)=M_1r^{(2A-1)a}$$

from Eq. (16)

$$\left(\frac{dr}{dz}\right)^2\frac{1}{rm(r)}\frac{dn(r)}{dr}=b^2 \quad \text{i.e.,} \quad n(r)=C+\frac{b^2aM_1A_1^{\,2}}{2A+1}r^{(2A+1)a}$$

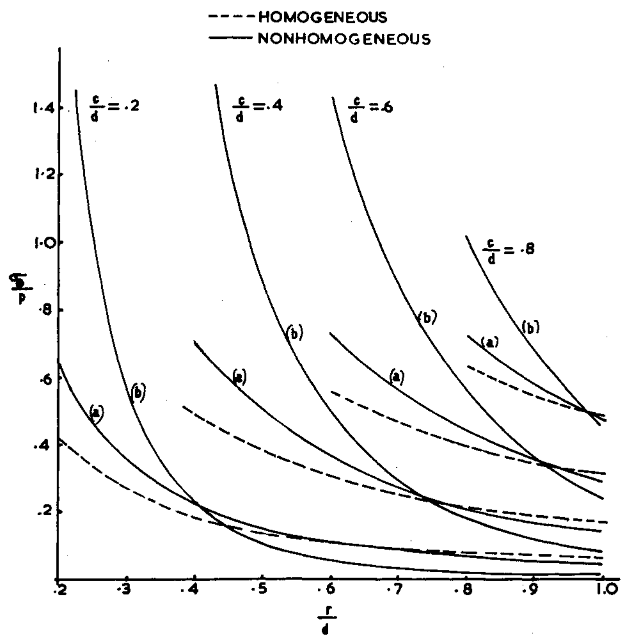

Fig. 2 Tangential stresses for cases a and b.

and from Eq. (17)

$$\left(\frac{dr}{dz}\right)^2 \frac{1}{r^2}\left\{1 + \frac{l(r)}{m(r)}\right\} = n^2 \quad \text{i.e., } l(r) = n^2 M_1 A_1{}^2 a^2 r^{(2A+1)a} - M_1 r^{(2A-1)a}$$

where A_1, M_1, C, a, b, n are all arbitrary constants. Then the solution of Eq. (3) is

$$df(r)/dr = r^{-(a/2)(2A-1)}[EK_v(\gamma A_1 r^a) + F(\gamma A_1 r^a)] \qquad (22)$$

where E and F are unknown constants, K_v and I_v are the modified Bessel Functions and v and γ are obtained from Eq. (18) as $\gamma^2 = n^2\beta^2 - \alpha b^2$, $v^2 = \frac{1}{4} - A(1-A) - \beta/a^2$. Now, evaluating the constants by the boundary conditions (20) and (21), the expressions of the stresses are easily obtained from Eq. (4). Taking the arbitrary constants as $A_1 = 1$, $M_1 = 1$, $C = 0$, $n^2 = 1$, $b^2 = 1$, $a = 1$, $A = 0.35566835$ and using the values of the material constants for Boron Epoxy, i.e., $E_\theta = E_z = 3.6 \times 10^6$ lb/in.^{2}, $E_r = 30.8 \times 10^6$ lb/in.^{2}, $v_{zr} = v_{\theta r} = 0.0421$, $v_{z\theta} = 0.3$, and $t/d = \frac{1}{40}$, $E/(1-v^2)E_z = 50$ for the elastic casing, we have, for $\tilde{r} = r/d$, $\tilde{c} = c/d$,

$$\sigma_r = -(P\tilde{c}^{x+1}/\Delta\tilde{r}^{x+1})\{B\gamma\tilde{\alpha}_{22}[I_1(\gamma)K_0(\gamma\tilde{r}) + K_1(\gamma)I_0(\gamma\tilde{r})] + [B(\tilde{\alpha}_{12} - x\tilde{\alpha}_{22}) + 1][K_0(\gamma\tilde{r})I_0(\gamma) - K_0(\gamma)I_0(\gamma\tilde{r})]\}$$

and

$$\sigma_\theta = (P\tilde{c}^{x+1}/\Delta\tilde{r}^{x+1})\{[B\gamma\tilde{\alpha}_{22}I_1(\gamma) + \{B(\tilde{\alpha}_{12} - x\tilde{\alpha}_{22}) + 1\}I_0(\gamma)] \times [\gamma\tilde{r}K_1(\gamma\tilde{r}) - xK_0(\gamma\tilde{r})] - [B\gamma\tilde{\alpha}_{22}K_1(\gamma) - \{B(\tilde{\alpha}_{12} - x\tilde{\alpha}_{22}) + 1\} \times K_0(\gamma)][\gamma\tilde{r}I_1(\gamma\tilde{r}) - xI_0(\gamma\tilde{r})]\}$$

$$\Delta = B\gamma\tilde{\alpha}_{22}\{K_0(\gamma\tilde{c})I_1(\gamma) + I_0(\gamma\tilde{c})K_1(\gamma)\} + \{B(\tilde{\alpha}_{12} - x\tilde{\alpha}_{22}) + 1\}\{K_0(\gamma\tilde{c})I_0(\gamma) - I_0(\gamma\tilde{c})K_0(\gamma)\}$$

where $B = 1.25$, $\gamma = 0.558171$, $\tilde{\alpha}_{22} = E_z\alpha_{22} = 0.91$, $\tilde{\alpha}_{11} = E_z\alpha_{11} = 0.11511$, $\tilde{\alpha}_{12} = E_z\alpha_{12} = -0.1684$, $\beta = 0.1265$, $\alpha = -0.185055$, $x = 0.355668$. In Figs. 1 and 2, the results of the analysis are presented.

b) Nonhomogeneities based on Helmholtz's equation

Let

$$P_1 = \bar{a}^2,\ P_2 = P_3 = 0; \quad Q_1 = -g^2,\ Q_2 = Q_3 = 0;$$
$$R_1 = b^2,\ R_2 = R_3 = 0; \quad K_1 = n^2,\ K_2 = K_3 = 0$$

As before, we obtain from Eqs. (14–17)

$$\left.\begin{aligned} z(r) = \log(A_1 r^a), \quad \alpha_{22} = \bar{\alpha}_{22}M_1 r^{2ag} \\ \alpha_{12} = \bar{\alpha}_{12}\left[C + \frac{a^2b^2M_1}{2ag}r^{2ag}\right], \quad \alpha_{11} = \bar{\alpha}_{11}[(n^2a^2 - 1)M_1 r^{2ag}] \end{aligned}\right\} \qquad (23)$$

and the solution of Eq. (3) is

$$df/dr = r^{-ag}[Er^{a(\beta n2 + g2 - xb2 - \beta a - 2)1/2} + Fr^{-a(\beta n2 + g2 - xb2 - \beta a - 2)1/2}]$$

where C, A_1, M_1, g, a, b, and n are all arbitrary constants. Evaluating the expressions of the stresses as before and taking the arbitrary constants as $g = b^2 = M_1 = a = 1$, $n^2 = 2$, $C = 0$, the results are presented in Figs. 1 and 2 for the material constants considered in (a).

Special case Putting $2g = m$, $a = 1$, $b^2 = m$, $n^2 = 2$, $M_1 = 1$, $C = 0$ in Eq. (23), we have $\alpha_{22} = \bar{\alpha}_{22}r^m$, $\alpha_{12} = \bar{\alpha}_{12}r^m$ and $\alpha_{11} = \bar{\alpha}_{11}r^m$ which is the case of Bieniek et al.[1] Then the solution of Eq. (3) is given by

$$df(r)/dr = r^{-m/2}[Er^{1/2\{m2 + 4(\beta - xm)\}1/2} + Fr^{-1/2\{m2 + 4(\beta - xm)\}1/2}]$$

which is exactly the same as that of Bieniek et al.[1] [cf. Eq. (25)].

5. Conclusions

The preceding analysis can be considered for the solid rocket motor system as a first approximation. Actually, solid rocket motor consists of solid propellant grains surrounded by a thin rubberlike insulator with outer surface reinforced by an elastic cylindrical shell and is subjected to internal pressure. Therefore, for the short-time pressurization, it can be taken as a thick anisotropic elastic cylinder of a material of properties varying along the radius as a first approximation with the boundary conditions considered in the analysis. Moreover, thermal stresses are excluded for the present analysis, as they are of secondary importance for the solid rocket motor which has a very short burning period. However, the author wishes to consider this effect in another Note.

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Quasi-Steady Gas-Phase Flame Theory in Unsteady Burning of a Homogeneous Solid Propellant

F. A. Williams*

University of California at San Diego, La Jolla, Calif.

Nomenclature

B = pre-exponential constant in Arrhenius rate expression
c_p = specific heat at constant pressure
D = diffusion coefficient
E = activation energy for gas-phase reaction
E' = nondimensional activation energy, Ec_p/RQ
L = sum of heat of gasification and heat conducted into interior of solid, both per unit mass of solid gasified
l = nondimensional heat flux into solid, L/Q
m = mass flux
n = reaction order
p = pressure
Q = heat released by the gas-phase reaction per unit mass of reactant consumed
R = universal gas constant
T = temperature
T_f = flame temperature
T_s = temperature of the solid surface
x = distance normal to propellant surface
x_f = distance between solid surface and gas-phase reaction zone, i.e. "flame thickness"
Y = reactant mass fraction
y = stretched nondimensional temperature, $\beta(\theta_f - \theta)$
α = diffusivity
β = nondimensional activation energy, $EQ/Rc_p T_f{}^2$
Λ = burning-rate eigenvalue, Eq. (18)
λ = thermal conductivity
θ = nondimensional temperature, $c_p T/Q$
ρ = density of the gas

Introduction

In theoretical analyses of solid-propellant combustion instability, models are needed that describe the gas-phase flame with simplicity sufficient for inclusion in rather complex time-dependent calculations. To date, all such models that have been

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* Professor of Aerospace Engineering, Department of Applied Mechanics and Engineering Sciences.